

REVERSE MATHEMATICS AND A RAMSEY-TYPE KÖNIG'S LEMMA

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ABSTRACT. In this paper, we propose a weak regularity principle which is similar to both weak König's lemma and Ramsey's theorem. We begin by studying the computational strength of this principle in the context of reverse mathematics. We then analyze different ways of generalizing this principle.

1. INTRODUCTION

Weak König's lemma states that for any infinite binary tree $T \subseteq 2^{<\mathbb{N}}$, there is an infinite path p through T . When formalized in second order arithmetic, this theorem is denoted WKL_0 . There is a direct correspondence between a path p through a binary tree, which is a function $p : \mathbb{N} \rightarrow \{0, 1\}$, and the set $\{x : p(x) = 1\}$. With this in mind, we occasionally identify paths through trees, colorings of singletons, and subsets of \mathbb{N} .

Ramsey's theorem is a generalization of the pigeon-hole principle. Given $n \in \mathbb{N}$, let $[\mathbb{N}]^n$ denote the collection of size n subsets of \mathbb{N} . The infinite version of Ramsey's theorem says that for every $n, k \in \mathbb{N}$ and every function (called a coloring) $f : [\mathbb{N}]^n \rightarrow \{0, \dots, k-1\}$, there is an infinite set $H \subseteq \mathbb{N}$ which is given one color by f . In the context of second order arithmetic, this principle is denoted RT_k^n (with n, k as above).

Weak König's lemma and Ramsey's theorem can be thought of as asserting the existence of different types of regularity. Viewed topologically, weak König's lemma is essentially the statement “ $2^{\mathbb{N}}$ is compact.” This carries over to reverse mathematics, where WKL_0 is equivalent to many theorems about compactness (see [13]). For its part, Ramsey's theorem says that no matter how badly behaved a coloring is, it always has a sizable homogeneous set. In the words of T.S. Motzkin, absolute disorder is impossible.

In this paper, we study the computational and reverse mathematical strength of a regularity principle which combines features of weak König's lemma and Ramsey's theorem. We will refer to the statement “for each infinite binary tree T , there is an infinite set H homogeneous for a path through T ” as a Ramsey-type König's lemma. In statement 2, we give a formal definition of our Ramsey-type König's

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lemma, denoted RKL , in terms of finite strings. It is the novelty of this principle, rather than the complexity of the proofs, that is the main innovation of this paper.

We begin by showing that RKL is a nontrivial weakening of WKL_0 and of RT_2^2 . More formally, we show that SRT_2^2 implies RKL , that WKL_0 implies RKL , and that RKL implies DNR (unless specified, we always work over RCA_0). Applying results of [1] and [11], we conclude that RKL is strictly weaker than WKL_0 and SRT_2^2 .

In the remaining sections, we state analogs of RKL for trees generated by infinite sets of strings ($\text{RKL}^{(1)}$) and for arithmetically-definable trees ($\text{RKL}^{(<\omega)}$). We then study the strength of each principle, and obtain the surprising result that these stronger principles are more closely related to RT_2^2 than to WKL_0 .

We show that RT_2^2 implies $\text{RKL}^{(1)}$, and that $\text{RKL}^{(1)}$ implies SRT_2^2 . By the main results of [5] and [11], it follows that $\text{RKL}^{(1)}$ and WKL_0 are incomparable. We also show that RT_2^2 does not imply $\text{RKL}^{(<\omega)}$ and, by using a result of [11], we show that $\text{RKL}^{(<\omega)}$ does not imply WKL_0 . We leave open whether $\text{RKL}^{(<\omega)}$ implies RT_2^2 . We summarize our results in figure 1.

1.1. Working in second order arithmetic. We assume that the reader is familiar with the basic definitions and results of computability theory and reverse mathematics. For an introduction to reverse mathematics, see chapter I of [13]. For an introduction to computability theory, see part A of [14].

Some care is required to formalize RKL in RCA_0 . Our goal here is to study the computational complexity of the homogeneous set H , not of the path p . While there are computable trees T such that each path through T is reasonably complicated, there are paths p with relatively simple homogeneous sets H . We begin with definitions that allow us to say that H is homogeneous for *some* path through T without explicit reference to a path p .

Definition 1 (RCA_0). H is homogeneous for $\sigma \in 2^{<\mathbb{N}}$ with color $c \in \{0, 1\}$ if $\sigma(x) = c$ for each $x \in H$ s.t. $x < |\sigma|$. H is homogeneous for a path through T if $\exists c \in \{0, 1\}$ s.t. H is homogeneous for σ with color c for arbitrarily long $\sigma \in T$.

Statement 2 (RCA_0). RKL asserts that “for each infinite binary tree T , there is an infinite set H which is homogeneous for a path through T .”

Unless stated otherwise, all strings and trees we consider will be binary ($\{0, 1\}$ -valued). Given $\tau, \sigma \in 2^{<\mathbb{N}}$ we write $\tau \preceq \sigma$ if τ is an initial segment of σ . We write $\sigma \upharpoonright t$ (or $p \upharpoonright t$) to denote the initial segment of $\sigma \in 2^{<\mathbb{N}}$ (or $p \in 2^{\mathbb{N}}$) of length t .

2. REVERSE MATHEMATICS OF RKL

Theorem 3 (RCA_0). WKL_0 implies RKL .

Proof. Given an infinite binary tree T , let p be an infinite path through T . Note that $p : \mathbb{N} \rightarrow \{0, 1\}$ maps singletons into two colors. Applying RT_2^1 , which is provable in RCA_0 , yields a set H which is homogeneous for p . In particular, $p \upharpoonright t \in T$ and H is homogeneous for $p \upharpoonright t$ for each $t \in \mathbb{N}$. Thus H satisfies definition 1, as desired. \square

Definition 4 (RCA_0). A coloring $f : [\mathbb{N}]^2 \rightarrow \{0, 1\}$ is *stable* if for each x , there is some $t > x$ s.t. $(\forall y > t)[f(x, y) = f(x, t)]$. SRT_2^2 is the theorem “every stable coloring of pairs into two colors has an infinite homogeneous set.”

Theorem 5 (RCA_0). SRT_2^2 implies RKL .

Proof. Given an infinite tree T , we define a coloring $f : [\mathbb{N}]^2 \rightarrow \{0, 1\}$ as follows. For each y , let σ_y be the lexicographically least element of T of length y . For each $x < y$, define $f(x, y) = \sigma_y(x)$.

We now show that f is a stable coloring. Fix x , and let T^{ext} denote the strings in T that are extended by arbitrarily long strings in T . For each $\tau \in T - T^{ext}$ of length $x + 1$, there is a bound on the length of strings in T extending τ , so there is a least such bound s_τ . Note that s_τ is Δ_1^0 definable (with parameters) from τ . By Σ_1^0 induction, there is a uniform upper bound t for $\{s_\tau : \tau \in 2^{x+1} \wedge \tau \in T - T^{ext}\}$. By Π_1^0 induction, there is a lexicographically least element $\tau_{x+1} \in T^{ext}$ of length $x + 1$. Then for each $y > t$, $\sigma_y \upharpoonright (x + 1) = \tau_{x+1}$ hence $f(x, y) = \tau_{x+1}(x)$. In general, for each x , $(\exists t)(\forall y > t)[f(x, y) = \tau_{x+1}(x)]$. In other words, f is a stable coloring.

Suppose that H is homogeneous for f with color $c \in \{0, 1\}$. We now show that H is homogeneous for a path through T . Fix $t \in \mathbb{N}$. Because H is an infinite set, there is an element $y \in H$ with $y \geq t$. By the definition of f , $(\forall x < y)[\sigma_y(x) = f(x, y)]$. Because H is homogeneous for f with color c and because $y \in H$, $(\forall x < y)[x \in H \implies \sigma_y(x) = c]$. Then H is homogeneous for $\sigma_y \in T$ with color c . Since t is arbitrary and $|\sigma_y| \geq t$, we have satisfied definition 1. \square

Corollary 6 (RCA_0). *RKL does not imply SRT_2^2 or WKL_0 .*

Proof. By the main result of [11], SRT_2^2 does not imply WKL_0 . Because SRT_2^2 implies RKL, RKL cannot imply WKL_0 . Similarly, RKL cannot imply SRT_2^2 over RCA_0 because WKL_0 does not imply SRT_2^2 (by Theorem 3.3 of [12] and Theorem 10.5 of [1]). \square

We conclude our analysis of RKL by showing that it is not provable in RCA_0 , and by showing that RKL is strong enough to imply DNR. When $T \subseteq 2^{<\mathbb{N}}$ is a tree, $[T] \subseteq 2^{\mathbb{N}}$ will be the set of infinite paths through T . The following lemma follows from the proof of lemma 2 in [10].

Lemma 7 (Jockusch, [10]). *There is an infinite computable tree T such that for any $p \in [T]$ and for any $e \in \mathbb{N}$, if $|W_e| \geq e + 3$ then W_e is not homogeneous for p . In fact, for each $e \in \mathbb{N}$, there is a $t \in \mathbb{N}$ s.t. if $|W_e| \geq e + 3$ then W_e is not homogeneous for any string in T of length greater than t .*

A simple corollary is that $\text{RCA}_0 \not\models \text{RKL}$, via an ω -model. Adapting the proof of Theorem 2.3 from [8], we can obtain a slightly stronger result. We say that a function f is *diagonally non-computable* relative to X if $f(e) \neq \Phi_e^X(e)$ for each e s.t. $\Phi_e^X(e) \downarrow$. The principle DNR asserts that for any set parameter X , there is a function that is diagonally non-computable relative to X .

Theorem 8 (RCA_0). *RKL implies DNR.*

Proof. We work relative to a set parameter X . The proof of lemma 2 of [10] (lemma 7 above) works in RCA_0 . Let T be the tree defined in this proof. By RKL, there is a set H homogeneous for a path through T . Note that there is a Δ_1^0 definable function $g : \mathbb{N} \rightarrow \mathbb{N}$ such that $W_{g(e)}$ is the least $e + 3$ elements of H (in the \mathbb{N} order).

We now show that g is a fixed point free function. If $|W_e| < 2^{e+1}$, then $|W_{g(e)}| \neq |W_e|$. Suppose that $|W_e| \geq 2^{e+1}$. By the above lemma, there is some t s.t. W_e is not homogeneous for any $\sigma \in T$ of length greater than t . Because $W_{g(e)} \subset H$ and because H is homogeneous for a path through T , $W_{g(e)}$ is homogeneous for arbitrarily long $\sigma \in T$. In particular $W_e \neq W_{g(e)}$. In other words, g is fixed point

free, so can be used to give a Δ_1^0 definition for a DNR function (formalize V.5.8 of [14] in RCA_0). \square

Question 9. Does DNR imply RKL?

A number of principles are known to be stronger than DNR, such as ASRAM and ASRT_2^2 from [7]. Proving that one of these principles does not imply RKL would separate DNR from RKL.

3. TREES GENERATED BY SETS OF STRINGS

Definition 10. Given an infinite set of strings Σ , let T_Σ denote the downward closure of Σ . More formally $T_\Sigma = \{\tau : (\exists \sigma \in \Sigma)[\tau \preceq \sigma]\}$.

Statement 11 (RCA_0). $\text{RKL}^{(1)}$ asserts that “for each infinite set of strings Σ , there is a set H which is homogeneous for a path through T_Σ .”

Note that if Σ is an infinite computable set of strings, T_Σ is an infinite c.e. tree. In [6], Downey and Jockusch note that each infinite $\Pi_1^{0,\theta'}$ -class can be generated by a c.e. tree. We extend this slightly, to further motivate our definition of $\text{RKL}^{(1)}$.

Proposition 12. *If T is an infinite Π_2^0 tree, then there is an infinite computable set of strings Σ s.t. $[T] = [T_\Sigma]$. Furthermore, Σ can be taken to contain exactly one string of each length.*

Proof. It suffices to consider Σ_1^0 trees. To see this, suppose that T is Π_2^0 . Then there is a formula ϕ which is Δ_1^0 s.t. $\tau \in T \leftrightarrow (\forall y)(\exists z)\phi(\tau, y, z)$. Using the Δ_1^0 formula $\psi(\tau, \hat{z}) =_{\text{def}} (\forall x, y \leq |\tau|)(\exists z < \hat{z})\phi(\tau \upharpoonright x, y, z)$, we can define a Σ_1^0 tree S by $\tau \in S \leftrightarrow (\exists \hat{z})\psi(\tau, \hat{z})$. Then $[S] = \{f : (\forall w)(\exists \hat{z})\psi(f \upharpoonright w, \hat{z})\} = \{f : (\forall x)(\forall y)(\exists z)\phi(f \upharpoonright x, y, z)\} = [T]$, so we may work with S instead.

Given a Σ_1^0 tree T , fix a computable enumeration $\{T_s\}$ of T . If necessary, we computably modify the enumeration to ensure that no τ enters T_s until $s \geq |\tau|$ and that exactly one string enters T at each stage. We computably enumerate the elements of Σ in increasing order. At stage $s > 0$, find $\tau \in T_s - T_{s-1}$, take one $\sigma \succeq \tau$ with $|\sigma| = s$ (the specific choice is not important), and put σ into Σ . It is not difficult to show that $T \subseteq T_\Sigma$ and that $T_\Sigma^{\text{ext}} \subseteq T^{\text{ext}}$. It follows that $[T] = [T_\Sigma]$. \square

We now examine the strength of $\text{RKL}^{(1)}$.

Theorem 13 (RCA_0). RT_2^2 implies $\text{RKL}^{(1)}$.

Proof. Fix an infinite set of strings Σ . For each y , let $l \geq y$ be the length of the shortest string in Σ of length at least y . Let σ_y be the lexicographically least string in Σ of length l . We now define a coloring $f : [\mathbb{N}]^2 \rightarrow \{0, 1\}$ as before. For each $x < y \in \mathbb{N}$, set $f(x, y) = \sigma_y(x)$. Note that f is Δ_1^0 -definable.

By RT_2^2 , there is an infinite set H homogeneous for f with color $c \in \{0, 1\}$. We claim that H is homogeneous for a path through T_Σ .

Fix $t \in \mathbb{N}$. Because H is infinite, there is some $y \in H$ with $y \geq t$. By definition of f , $(\forall x < y)[f(x, y) = \sigma_y(x)]$. Because H is homogeneous for f with color c and because $y \in H$, $(\forall x < y)[x \in H \implies \sigma_y(x) = c]$. In other words, H is homogeneous for $\sigma_y \upharpoonright y$ with color c . Because $\sigma_y \upharpoonright y \in T_\Sigma$ and because $y \geq t$ with t arbitrary, H is homogeneous for a path through T_Σ (in the sense of definition 1). \square

Remark 14. The coloring defined in the proof of theorem 13 is not (necessarily) stable because it is defined in terms of Σ , and not in terms of T . For example, suppose that $\sigma(0) = 0$ for even length strings $\sigma \in \Sigma$, and that $\sigma(0) = 1$ for odd length strings. Then $\lim_y f(0, y)$ does not exist.

There is a natural correspondence between computable colorings $f : [\mathbb{N}] \rightarrow \{0, 1\}$ and computable sets $\Sigma \subset 2^{<\mathbb{N}}$ that contain exactly one string of each length. Given f , simply define $\Sigma = \{\sigma_y : \sigma_y \in 2^y \wedge (\forall x < y)[\sigma_y(x) = f(x, y)]\}$. Using the induced tree T_Σ , it is not difficult to show that $\text{RKL}^{(1)}$ implies SRT_2^2 over $\text{RCA}_0 + \text{B}\Sigma_2^0$. Yokoyama was able to eliminate the use of $\text{B}\Sigma_2^0$ by introducing the following principle.

Statement 15 (RCA_0). P_2^2 asserts that “for any Π_2^0 -definable subsets A_0, A_1 of \mathbb{N} s.t. $A_0 \cup A_1 = \mathbb{N}$, there exists an infinite set $H \in S(\mathcal{M})$ s.t. $H \subseteq A_0$ or $H \subseteq A_1$.”

The principle P_2^2 is particularly useful because it implies the better understood principle D_2^2 . This is a special instance of the principle D_2^n , which we will return to in the next section.

Statement 16 (RCA_0). For each $n \in \omega$, D_2^n asserts that “for each Δ_n^0 -definable subset A of \mathbb{N} , there exists an infinite set $H \in S(\mathcal{M})$ s.t. $H \subseteq A$ or $H \subseteq \bar{A}$.”

Theorem 17 (Cholak, Chong, Jockush, Lempp, Slaman, Yang [1, 4]). *Over RCA_0 , D_2^2 is equivalent to SRT_2^2 .*

Theorem 18 (Yokoyama [15]). $\text{RKL}^{(1)}$ implies P_2^2 , and hence SRT_2^2 , over RCA_0 .

Proof. Let $\mathcal{M} = (\mathbb{N}, S(\mathcal{M})) \models \text{RCA}_0 + \text{RKL}^{(1)}$ and suppose that A_0, A_1 are Π_2^0 -definable subsets of \mathbb{N} s.t. $A_0 \cup A_1 = \mathbb{N}$. We will define a Δ_1^0 function $f : [\mathbb{N}]^2 \rightarrow \{0, 1\}$ s.t. if $f(x, y) = i$ for infinitely many y , then $x \in A_i$.

Fix two Σ_0^0 formulas $\theta_i(x, m, n)$ s.t. $x \in A_i \iff (\forall m)(\exists n)\theta_i(x, m, n)$. Using these formulas, we define a helper function:

$$h(x, y) = (\mu z)[(\forall m < y)(\exists n < z)[\theta_0(x, m, n)] \vee (\forall m < y)(\exists n < z)[\theta_1(x, m, n)]].$$

Clearly, h is a Δ_1^0 function.

We must verify in RCA_0 that h is total. Fix $x \in \mathbb{N}$ arbitrary. Then $x \in \mathbb{N} = A_0 \cup A_1$, so $x \in A_0$ or $x \in A_1$. So $(\forall m)(\exists n)\theta_0(x, m, n)$ or $(\forall m)(\exists n)\theta_1(x, m, n)$. Let $y \in \mathbb{N}$ be arbitrary. Suppose $x \in A_i$. Then $(\forall m)(\exists n)\theta_i(x, m, n)$, so clearly $(\forall m < y)(\exists n)\theta_i(x, m, n)$. By $\text{B}\Sigma_1^0$, there is a z_i s.t. $(\forall m < y)(\exists n < z_i)\theta_i(x, m, n)$. Thus, h will find a least z s.t. the desired condition holds.

Define $f(x, y) = 0$ if $(\forall m < y)(\exists n < h(x, y))[\theta_0(x, m, n)]$, and $f(x, y) = 1$ otherwise. Clearly, f is total since h is total, and f is Δ_1^0 since h is total and Δ_1^0 . If $f(x, y) = i$ for infinitely many y , then our defn of $h(x, y)$ confirms that $x \in A_i$.

Using f , let $\Sigma = \{\sigma_y : \sigma_y \in 2^y \wedge (\forall x < y)[\sigma_y(x) = f(x, y)]\}$ and define T_Σ as before. Take H homogeneous for a path through T_Σ with some color $c \in \{0, 1\}$.

Let $x \in H$ be arbitrary. By definition of “homogeneous for a path through T_Σ with color c ,” there are infinitely many y s.t. $f(x, y) = \sigma_y(x) = c$. By our choice of f , this means that $x \in A_c$. In other words, $H \subseteq A_c$ is the desired infinite set. \square

Question 19 (Yokoyama [15]). Does D_2^2 imply P_2^2 ? Does P_2^2 imply $\text{RKL}^{(1)}$?

Corollary 20. $\text{RKL}^{(1)}$ is incomparable with WKL_0 over RCA_0

Proof. Because WKL_0 does not imply SRT_2^2 (Theorem 3.3 of [12] and Theorem 10.5 of [1]), WKL_0 does not imply $\text{RKL}^{(1)}$. By the main result of [11], RT_2^2 does not imply WKL_0 , so $\text{RKL}^{(1)}$ cannot imply WKL_0 . \square

Remark 21. Using the above arguments, we can rephrase RT_2^2 as the statement “for each Σ which contains exactly one string of each length, there is an infinite H which is homogeneous (with fixed color c) for each $\sigma \in \Sigma$ s.t. $|\sigma| \in H$.”

Question 22. Does $\text{RKL}^{(1)}$ imply COH , CAC , ADS or RT_2^2 ? One implication holds if and only if all implications hold. Does SRT_2^2 imply $\text{RKL}^{(1)}$?

4. ARITHMETICALLY-DEFINABLE TREES

Statement 23 (RCA_0). $\text{RKL}^{(<\omega)}$ is the axiom scheme which, for each arithmetic formula ϕ , asserts that “if ϕ defines a tree T containing arbitrarily long strings, there is an infinite set H which is homogeneous for a path through T .”

Theorem 24. Over RCA_0 , we have the following strict implications: $\text{ACA}_0 \implies \text{RKL}^{(<\omega)} \implies \text{RKL}^{(1)} \implies \text{RKL}$.

The implications are immediate. We have already seen that the third implication is strict. We now show that the first two implications are also strict. We first use the following result from [11] to separate $\text{RKL}^{(<\omega)}$ from ACA_0 .

Theorem 25 (Liu, [11]). For every $C \not\geq \emptyset$ and every coloring $p : \mathbb{N} \rightarrow \{0, 1\}$, there exists an infinite set H homogeneous for p such that $H \oplus C \not\geq \emptyset$.

Corollary 26. There is an ω -model of $\text{RKL}^{(<\omega)}$ where WKL_0 fails.

Proof. To build an ω -model $\mathcal{M} = (\omega, S(\mathcal{M}))$ of $\text{RKL}^{(<\omega)}$, we begin with $S(\mathcal{M}) = \text{REC}$ and add sets to $S(\mathcal{M})$.

The general strategy for creating a model of $\text{RKL}^{(<\omega)}$ uses a list of the infinite trees which are arithmetically-definable from any set $X \in S(\mathcal{M})$. For each $i \in \mathbb{N}$, we must ensure that there is some finite stage s where we select a path p through the i^{th} tree T , where we select an infinite set H_s homogeneous for p , and where we add H_s to $S(\mathcal{M})$ and close downward under \leq_T . To ensure that $\mathcal{M} \not\models \text{WKL}_0$, we use Theorem 25 to select H_s s.t. $H_s \oplus \bigoplus_{j \leq s-1} H_j \not\geq \emptyset$.

It is possible that adding the set H_s to $S(\mathcal{M})$ causes new sets to become arithmetically-definable with parameters from $S(\mathcal{M})$. Therefore, each time we add H_s to $S(\mathcal{M})$, we create a new list containing the trees arithmetically-definable from $\bigoplus_{i \leq s} H_i$. We dovetail the lists, eventually running the general strategy for each tree in each list. In the limit, we obtain $\mathcal{M} \models \text{RKL}^{(<\omega)} + \neg \text{WKL}_0$. \square

Corollary 27 (RCA_0). $\text{RKL}^{(<\omega)}$ does not imply WKL_0 .

We separate $\text{RKL}^{(<\omega)}$ from $\text{RKL}^{(1)}$ with an ω -model by the following observation.

Lemma 28. For each n , no model of $\text{RKL}^{(<\omega)}$ is bounded by \emptyset^n .

Proof. By the proof of lemma 7 relativized to $X = \emptyset^n$, we obtain an \emptyset^n -computable infinite tree T s.t. no infinite set $W_e^{\emptyset^n}$ is homogeneous for a path through T . Since each \emptyset^n -computable set is $W_e^{\emptyset^n}$ for some e , it follows that no infinite \emptyset^n -computable set is homogeneous for a path through T . \square

Proposition 29. RT_2^2 does not imply $\text{RKL}^{(<\omega)}$ over RCA_0 . Consequently, $\text{RKL}^{(1)}$ does not imply $\text{RKL}^{(<\omega)}$ over RCA_0 .

Proof. By the previous lemma, there is no model of $\text{RKL}^{(<\omega)}$ which is bounded by \emptyset^2 . By Theorem 3.1 of [1], there is an ω -model of RT_2^2 consisting of only low_2 sets. This model is bounded by \emptyset^2 so is not a model of $\text{RKL}^{(<\omega)}$. \square

Question 30. Does $\text{RKL}^{(<\omega)}$ imply COH over RCA_0 ? Equivalently, does $\text{RKL}^{(<\omega)}$ imply RT_2^2 over RCA_0 ?

4.1. Subsets, co-subsets, and trees. There is a close relationship between finding subsets/cosubsets of a fixed set, and finding sets that are homogeneous for a path through a fixed tree.

Statement 31 (RCA_0). We define $D_2^{<\omega}$ to be the axiom scheme which asserts D_2^n for each $n \in \omega$.

Proposition 32 (RCA_0). $\text{RKL}^{(<\omega)}$ implies $D_2^{<\omega}$.

Proof. Let $\mathcal{M} = (\mathbb{N}, S(\mathcal{M})) \models \text{RCA}_0 + \text{RKL}^{(<\omega)}$ and suppose that A is a Δ_n^0 -definable subset of \mathbb{N} . We give a Π_n^0 definition for a tree T as follows. Given $\tau \in 2^{<\mathbb{N}}$, we say that $\tau \in T$ if and only if $(\forall x < |\tau|)[\tau(x) = 1 \text{ if and only if } x \in A]$.

By $\text{RKL}^{(<\omega)}$, there is a set $H \in S(\mathcal{M})$ which is homogeneous for arbitrarily long strings in T with color $c \in \{0, 1\}$. Note that the only strings in T are initial segments of χ_A , so H is homogeneous for χ_A with color c . Then $H \subseteq A$ if $c = 1$, and $H \subseteq \bar{A}$ if $c = 0$, as desired. \square

Remark 33. For ω -models, the reverse implication also holds.

Question 34. Does $D_2^{<\omega}$ imply $\text{RKL}^{(<\omega)}$ over RCA_0 ?

By results of [1], SRT_2^2 implies $\text{B}\Sigma_2^0$.

Question 35. Are there first order consequences of $\text{RKL}^{(<\omega)}$ beyond $\text{B}\Sigma_2^0$?

Chong, Slaman, and Yang have recently announced a proof that D_2^2 does not imply COH over RCA_0 [3].

Question 36. Does D_2^n imply COH for any $n \in \omega$?

Theorem 2.1 of [9] gives another way to state this question for ω -models.

Question 37. Is there any arithmetically-definable $f : \mathbb{N} \rightarrow \{0, 1\}$ such that any set H homogeneous for f satisfies $H' \gg \emptyset'$?

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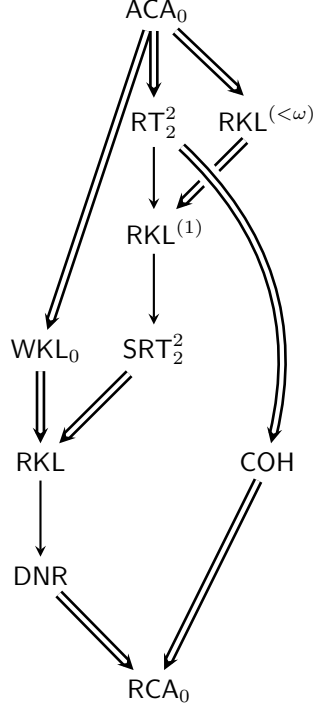


FIGURE 1. A summary of the principles considered.

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